ON A DEGENERATE PARABOLIC EQUATION ARISING IN PRICING OF ASIAN OPTIONS

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Abstract. We study a certain one dimensional, degenerate parabolic partial differential equation with a boundary condition which arises in pricing of Asian options. Due to degeneracy of the partial differential operator and the non-smooth boundary condition, regularity of the generalized solution of such a problem remained unclear. We prove that the generalized solution of the problem is indeed a classical solution.

1. Introduction and Main result

In [5], Večeř proposed a unified method for pricing Asian options, which lead to a simple one-dimensional partial differential equation

$$u_t + \tfrac{1}{2}\left(x - e^{-\int_0^t d\nu(s)}q(t)\right)^2 \sigma^2 u_{xx} = 0 \tag{1.1}$$

with the boundary condition

$$u(T,x) = (x-K_1)_+ := \max(x-K_1,0). \tag{1.2}$$

Here, $\nu(t)$ is the measure representing the dividend yield, σ is the volatility of the underlying asset, $q(t)$ is the trading strategy given by

$$q(t) = \exp\left\{-\int_t^T d\nu(s)\right\}\cdot\int_t^T \exp\left\{-r(T-s)+\int_s^T d\nu(\tau)\right\} d\mu(s),$$

where r is the interest rate and $\mu(t)$ represents a general weighting factor. In the fixed strike Asian call option, we have $K_1 = 0$ in the boundary condition (1.2); see [4, 5] for details. If we assume that $d\mu(t) = \rho(t)\,dt$ for some $\rho \in L^\infty([0,T])$ satisfying $0 < \rho_0 \leq \rho(t)$, then it is readily seen that

$$e^{-\int_0^t d\nu(s)}q(t) = c\int_t^T \exp\left\{-r(T-s)+\int_s^T d\nu(\tau)\right\} d\mu(s) \quad \left(c = e^{-\int_0^T d\nu(s)} > 0\right)$$

is a monotone decreasing Lipschitz continuous function. We are thus lead to consider the following one-dimensional parabolic PDE

$$u_t + \tfrac{1}{2}(b(t)-x)^2\, u_{xx} = 0 \tag{1.3}$$

in $H_T := (0,T)\times\mathbb{R}$ with the boundary condition

$$u(T,x) = x_+, \tag{1.4}$$

where $b(t)$ is a Lipschitz continuous function defined on $[0,T]$ such that $b(T)=0$ and

$$m_1 \leq -b'(t) \leq m_2, \ \text{ for a.e. } \ t\in(0,T) \ \text{ for some } \ m_1, m_2 > 0. \tag{1.5}$$

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In this article we are mainly concerned with regularity of the (generalized) solution $u(t,x)$ of the problem (1.3), (1.4). It is a rather nontrivial task to show that the problem (1.3), (1.4) has a solution in the classical sense. First of all, it should be noted that even though the coefficient which appears in (1.3) is Lipschitz continuous, the classical approach based on Schauder theory is not applicable here, for the operator in (1.3) becomes degenerate along the curve $x = b(t)$. Nevertheless, it is possible to show that the problem (1.3), (1.4) admits the "probabilistic" solution: Let

$$u(t,x) := \mathbb{E} f(X_T(t,x)), \tag{1.6}$$

where $f(x) := x_+$ and X_s is the stochastic process which satisfies, for $t \in [0,T]$ and $x \in \mathbb{R}$,

$$\begin{cases} dX_s(t,x) = (b_s - X_s(t,x))\, dw_s, \quad s \geq t, \quad (\, b_s = b(s)\,) \\ X_t(t,x) = x. \end{cases} \tag{1.7}$$

It is known that such a process X_t exists and that if f is twice continuously differentiable, then $u(t,x)$ given by (1.6) is a classical solution of (1.3) in H_T (i.e., $u(t,x)$ is continuously differentiable with respect t and twice continuously differentiable with respect to x in H_T and satisfies (1.3) there) with the boundary condition $u(T,x) = f(x)$; see e.g. [2]. Unfortunately, $f(x) = x_+$ is not twice continuously differentiable and the above method is not directly applicable here. On the other hand, it should be also noted that if $b(t)$ is smooth enough and $b'(t) \neq 0$ everywhere, then the differential operator in (1.3) satisfies Hörmander's conditions for hypoellipticity (see [1]). Therefore, in this case, it is not hard to see that $u(t,x)$ given by (1.6) becomes a classical solution of the problem (1.3), (1.4). However, Hörmander's theorem is not available under a mere assumption that $b(t)$ is a Lipschitz continuous function satisfying (1.5).

The main goal of this article is to present a technique to prove that the generalized solution $u(t,x)$ of the problem (1.3), (1.4) is indeed a classical solution. Let us now state our main theorem.

Theorem 1.8. *For $t \in [0,T]$ and $x \in \mathbb{R}$, let $X_s = X_s(t,x)$ be the stochastic process which satisfies* (1.7) *and let $u(t,x)$ be defined as in* (1.6) *with $f(x) := x_+$. Then $u(t,x)$ is a classical solution of the equation* (1.3) *in $H_T = (0,T) \times \mathbb{R}$ satisfying the boundary condition* (1.4)*.*

The organization of this paper is as follows. In Sec. 2, we introduce some notations and present a preliminary lemma which will be used in the proof of the main result. In Sec. 3, we give the proof of our main result, Theorem 1.8. An outline of the proof is as follows. We first split $u = u_1 + u_2$, where u_i are the probabilistic solutions of (1.3) satisfying $u_i(T,x) = f_i(x)$ with $f_1(x) = x$ and $f_2(x) = (-x)_+$. It can be readily seen that u_1 is a classical solution of (1.3) in H_T. Next, we show that $u_2 \equiv 0$ in the set $\{(t,x) \in H_T : x \geq b(t)\}$. Then, by using a suitable rescaling and the lemma in Sec. 2, we show that u_2 decays very rapidly to zero near the curve $x = b(t)$. This is the key point of the proof. Then, we apply the interior Schauder estimates to u_2 to conclude that $\partial_t u_2$, $\partial_x u_2$, and $\partial_{xx} u_2$ all decay rapidly to zero near the curve $x = b(t)$, from which we will be able to complete the proof. Finally, In Sec. 4, we reformulate the key lemma of the proof in more general settings, in the hope that this technique might be useful to some other problems as well.

2. Notations and preliminaries

2.1. **Some notations.** We introduce some notations which will be used in the proof. We define the parabolic distance between the points $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$ as

$$|z_1 - z_2|_p := \max(\sqrt{|t_1 - t_2|}, |x_1 - x_2|).$$

Let $\alpha \in (0,1)$ be a fixed constant. If u is a function in a domain $Q \subset \mathbb{R}^2$, we denote

$$[u]_{\alpha/2,\alpha;Q} = \sup_{\substack{z_1 \neq z_2 \\ z_1, z_2 \in Q}} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|_p^\alpha}, \quad |u|_{0;Q} = \sup_Q |u|,$$
$$|u|_{\alpha/2,\alpha;Q} = |u|_{0;Q} + [u]_{\alpha/2,\alpha;Q}.$$

By $C^{\alpha/2,\alpha}(Q)$ we denote the space of all functions for which $|u|_{\alpha/2,\alpha;Q} < \infty$. We also introduce the space $C^{1+\alpha/2,2+\alpha}(Q)$ as the set of all functions u defined in Q for which both

$$[u]_{1+\alpha/2,2+\alpha;Q} := [u_t]_{\alpha/2,\alpha;Q} + [u_{xx}]_{\alpha/2,\alpha;Q} < \infty \quad \text{and}$$
$$|u|_{1+\alpha/2,2+\alpha;Q} := |u|_{0;Q} + |u_x|_{0;Q} + |u_t|_{0,Q} + |u_{xx}|_{0,Q} + [u]_{1+\alpha/2,2+\alpha;Q} < \infty.$$

The function space $C^{1,2}(Q)$ denotes the set of all functions defined in Q for which

$$|u|_{0;Q} + |u_x|_{0;Q} + |u_t|_{0,Q} + |u_{xx}|_{0,Q} < \infty.$$

We say $u \in C^{1+\alpha/2,2+\alpha}_{loc}(Q)$ if $u \in C^{1+\alpha/2,2+\alpha}(Q')$ for all compact set $Q' \Subset Q$ and similarly, $u \in C^{1,2}_{loc}(Q)$ if $u \in C^{1,2}(Q')$ for all compact set $Q' \Subset Q$.

2.2. **A lemma on Gaussian estimates.** Let $R > 0$ be fixed and $g(x)$ be a continuous function defined on $[-R, R]$ satisfying $1/2 \le g(x) \le 3/2$ for $x \in [-R, R]$. We denote

$$Q := \{(t,x) \in \mathbb{R}^2 : 0 < t < 2, |x| < R\},$$
$$\Omega := \{(t,x) \in Q : t > g(x)\}, \quad \Sigma := \{(t,x) \in Q : t = g(x)\}.$$

Lemma 2.1. *Let Ω and Σ be defined as above and let $a(t,x)$ be a function satisfying*

$$0 \le a(t,x) \le 1, \quad \forall (t,x) \in \Omega. \tag{2.2}$$

Assume that $u \in C^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega})$ and satisfies

$$\begin{cases} Lu := u_t - a(t,x) u_{xx} = 0 & in \quad \Omega \\ u = 0 & on \quad \Sigma. \end{cases}$$

Then, we have the following estimate:

$$|u|_{0;\Omega'} \le (16/\sqrt{2\pi}) R^{-1} e^{-R^2/32} |u|_{0;\Omega}, \quad \textit{where } \Omega' := \{(t,x) \in \Omega : |x| < R/2\}. \tag{2.3}$$

Proof. By changing $u \to u/|u|_{0;\Omega}$, we may assume $|u|_{0;\Omega} = 1$. Let $\Phi(t,x)$ be the fundamental solution of the heat equation in $(0,\infty) \times \mathbb{R}$; i.e.,

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

let $v(t,x)$ be a function on $(0,\infty) \times \mathbb{R}$ defined by

$$v(t,x) = 2 \int_E \Phi(t, x-y)\, dy, \quad \text{where } E := \bigcup_{j \in \mathbb{Z}} ((4j+1)R, (4j+3)R). \tag{2.4}$$

Denote $D = \{(t,x) \in \mathbb{R}^2 : t > 0,\ |x| < R\}$. From (2.4), it follows that $v \ge 0$ and satisfies

$$\begin{cases} v_t - v_{xx} = 0 & \text{in} \quad D, \\ v = 0 & \text{on} \quad \partial_t D := \{(t,x) \in \mathbb{R}^2 : t = 0,\ |x| < R\}, \\ v = 1 & \text{on} \quad \partial_x D := \{(t,x) \in \mathbb{R}^2 : t > 0,\ |x| = R\}. \end{cases} \tag{2.5}$$

Moreover, by the comparison principle, we see that $v(t,x) \le v(t+h,x)$ in D for any $h > 0$, and thus it follows that

$$v_{xx} = v_t \ge 0 \quad \text{in} \quad D. \tag{2.6}$$

Then by using (2.2), we have

$$L(v \pm u) = Lv = v_t - a(t,x)v_{xx} \geq v_t - v_{xx} = 0 \quad \text{in} \quad \Omega.$$

Denote by $\partial_p \Omega$ the parabolic boundary of Ω (see e.g., [3] for its definition) and observe that $\Sigma' := \partial_p \Omega \setminus \Sigma \subset \partial_x D$. Then, by (2.5), we find (recall that we assume $|u|_{0;\Omega} = 1$)

$$v \pm u \geq 0 \quad \text{on} \quad \partial_p \Omega.$$

Therefore, by the maximum principle and (2.6), we have

$$|u(t,x)| \leq v(t,x) \leq v(2,x), \quad \forall (t,x) \in \Omega.$$

On the other hand, for $|x| < R/2$, we estimate $v(2,x)$ by

$$\begin{aligned} v(2,x) &= 2\int_E \Phi(2, x-y)\,dy \leq 4\int_{R-|x|}^{\infty} \Phi(2,y)\,dy \leq 4\int_{R/2}^{\infty} \Phi(2,y)\,dy \\ &\leq \frac{8}{\sqrt{8\pi}R}\int_{R/2}^{\infty} y e^{-y^2/8}\,dy = \frac{16}{\sqrt{2\pi}} R^{-1} e^{-R^2/32}. \end{aligned} \tag{2.7}$$

The lemma is proved. □

3. Proof of Theorem 1.8

For $t \in [0,T]$ and $x \in \mathbb{R}$, let $X_s = X_s(t,x)$ be the stochastic process which satisfies (1.7). It is well known that such a process X_t exists; see e.g., [2, Theorem V.1.1]. Denote

$$u_1(t,x) = \mathbb{E} f_1(X_T(t,x)), \quad u_2(t,x) = \mathbb{E} f_2(X_T(t,x)), \tag{3.1}$$

where $f_1(x) = x$ and $f_2(x) = (-x)_+$ so that $f(x) = f_1(x) + f_2(x)$. By [2, Theorem V.7.4], the function u_1 and its derivatives $\partial_t u_1$, $\partial_x u_1$, and $\partial_{xx} u_1$ are continuous in H_T and u_1 satisfies the equation (1.3) there. In other words, the function u_1 is a classical solution of (1.3) in H_T. Also, it is readily seen that $u_i \in C(\overline{H_T})$ $(i = 1,2)$. Therefore, it is clear that $u = u_1 + u_2$ satisfies the boundary condition (1.4).

Let us further analyze the function u_2. Once we prove that u_2 is also a classical solution of (1.3) in H_T, then we are done. Let $\{g_k\}_{k=1}^{\infty}$ be be smooth approximations of f_2, say obtained by using mollifiers, such that $g_k \to f_2$ uniformly. Denote

$$v_k(t,x) = \mathbb{E} g_k(X_T(t,x)).$$

Then by the same reasoning as above, the functions $\{v_k\}_{k=1}^{\infty}$ are classical solution of (1.3) in H_T. Note that by interior Schauder estimates, $C^{1+\alpha/2,2+\alpha}$-norm of v_k in any compact set belonging to $H_T \setminus \{(t,x) : x = b(t)\}$ is estimated through its supremum over a bounded domain containing the set. Since $g_k \to f_2$ uniformly, we also have $v_k \to u_2$ uniformly, and thus we get

$$u_2 \in C^{1+\alpha/2,2+\alpha}_{loc}(\Omega), \quad \text{where } \Omega := H_T \setminus \{(t,x) : x = b(t)\}$$

and satisfies the equation (1.3) in Ω.

Next, we claim that $u_2 \equiv 0$ in $\{(t,x) \in [0,T] \times \mathbb{R} : x \geq b(t)\}$. Note that the process

$$Y_s(t,x) := X_s(t,x) - b_s \qquad (b_s = b(s))$$

satisfies the following stochastic differential equation:

$$\begin{cases} dY_s(t,x) = -Y_s(t,x)\,dw_s - b'(s)\,ds, & s \geq t, \\ Y_t(t,x) = x - b(t). \end{cases} \tag{3.2}$$

The solution to (3.2) is unique and has a representation

$$Y_s = Y_t\, e^{w_t - w_s + \frac{1}{2}(t-s)} - \int_t^s e^{w_r - w_s + \frac{1}{2}(r-s)}\, b'(r)\,dr, \quad s \geq t.$$

Therefore, from the assumption $b' \le 0$, we conclude that $Y_s(t,x) \ge 0$ for all $s \ge t$ provided that $Y_t(t,x) = x - b(t) \ge 0$. In particular, we have $X_T(t,x) = X_T(t,x) - b(T) = Y_T(t,x) \ge 0$ if $x \ge b(t)$. Therefore, from (3.1) and the fact that $f_2 \equiv 0$ for $x \ge 0$, we find $u_2(x,t) = 0$ if $x \ge b(t)$. We have thus proved the claim that $u_2 \equiv 0$ in $\{(t,x) \in [0,T] \times \mathbb{R} : x \ge b(t)\}$.

Now, we will show that $u_2 \in C^{1,2}_{loc}(H_T)$. To comply with standard conventions in parabolic PDE theory, we make a change of variable $t \mapsto T - t$ and denote

$$v(t,x) := u_2(T-t,x) \quad \text{and} \quad \psi(t) := b(T-t). \tag{3.3}$$

By the observations made above, we have

$$v \in C(\overline{H}_T) \cap C^{1+\alpha/2,2+\alpha}_{loc}(H_T \setminus \Gamma), \quad \text{where } \Gamma := \{(t,x) \in H_T : x = \psi(t)\}, \tag{3.4}$$

and satisfies the equation

$$v_t - \tfrac{1}{2}(x-\psi(t))^2 v_{xx} = 0 \quad \text{in} \quad H_T \setminus \Gamma.$$

In order to show that $v \in C^{1,2}_{loc}(H_T)$, we need investigate the behavior of v near Γ. By (1.5), we find that $\phi := \psi^{-1}$ is defined on $[0,\ell]$, where $\ell := \psi(T)$, and satisfies

$$1/m_2 \le \phi'(x) \le 1/m_1, \ \text{ for a.e. } \ x \in (0,\ell).$$

In the rest of the proof, we use the following notation. For $z_0 = (t_0,x_0) \in \mathbb{R}^2$, we denote

$$\begin{aligned}
C_r(z_0) &= \{(t,x) \in \mathbb{R}^2 : |t-t_0| < r,\ |x-x_0| < (m_1/2)r\},\\
\mathcal{U}_r(z_0) &= C_r(z_0) \cap \{(t,x) \in H_T : x < \psi(t)\},\\
\mathcal{U}'_r(z_0) &= \{(t,x) \in \mathcal{U}_r(z_0) : |x-x_0| < (m_1/4)r\},\\
\Gamma_r(z_0) &= C_r(z_0) \cap \Gamma.
\end{aligned}$$

Lemma 3.5 (Key lemma)**.** *Let $z_0 = (t_0,x_0) = (t_0,\psi(t_0)) \in \Gamma$ and $r \in (0,1)$ be any number satisfying $\overline{C}_r(z_0) \subset D := (0,T) \times (0,\ell)$. Then, the function v defined as in* (3.3) *satisfies*

$$|v|_{0;\mathcal{U}'_r(z_0)} \le N_0 r^{1/2} e^{-k_0/r} |v|_{0;D}, \tag{3.6}$$

where $N_0 = N_0(m_1,m_2)$ and $k_0 = k_0(m_1,m_2) > 0$. Moreover, we have

$$r^{3/2}|v_x(t_0+r,x_0)| + r^3|v_{xx}(t_0+r,x_0)| + r|v_t(t_0+r,x_0)| \le N_1 r^{1/2} e^{-k_0/r} |v|_{0;D}, \tag{3.7}$$

where $N_1 = N_1(m_1,m_2)$.

Proof. Let T be a linear mapping defined by

$$T(t,x) := \left((t-t_0)/r, (x-x_0)/cr^{3/2}\right), \quad \text{where } c := (m_1+2m_2)/\sqrt{8}. \tag{3.8}$$

We shall denote $\Omega_r := T(\mathcal{U}_r(z_0))$, $\Sigma_r := T(\Gamma_r(z_0))$, and

$$Q_r := T(C_r(z_0)) = \{(t,x) \in \mathbb{R}^2 : |t| < 1,\ |x| < (m_1/2c)r^{-1/2}\}.$$

We also define the functions $w(t,x)$ and $a(t,x)$ on $\overline{Q}_r$ by

$$w(t,x) := v \circ T^{-1}(t,x) = v(t_0+rt, x_0+cr^{3/2}x), \tag{3.9}$$

$$a(t,x) := \frac{1}{2(cr)^2}\left(x_0 + cr^{3/2}x - \psi(t_0+rt)\right)^2. \tag{3.10}$$

Then $w \in C^{1+\alpha/2,2+\alpha}_{loc}(\Omega_r) \cap C(\overline{\Omega}_r)$ and satisfies

$$\begin{cases} Lw := w_t - a(t,x)w_{xx} = 0 & \text{in} \quad \Omega_r,\\ w = 0 \quad \text{on} \quad \Sigma_r, \end{cases} \tag{3.11}$$

Note that $a(t,x)$ satisfies the following inequalities in Q_r.

$$0 \le a(t,x) = \frac{1}{2(cr)^2}\left(x_0 + cr^{3/2}x - \psi(t_0+rt) + \psi(t_0) - x_0\right)^2 \le \frac{1}{2(cr)^2}\left(cr^{3/2}|x| + m_2 r|t|\right)^2 \le \frac{1}{8c^2}(m_1+2m_2)^2 = 1. \tag{3.12}$$

Also, observe that $\Sigma_r \subset \{(t,x)\in\mathbb{R}^2 : |t| < 1/2\}$. By (3.11) and (3.12), we may apply Lemma 2.1 to $u(t,x) = w(t+1,x)$ with $R = (m_1/2c)r^{-1/2}$ to conclude that

$$|w|_{0;\Omega'_r} \le N r^{1/2} e^{-k_0/r}\, |w|_{0;\Omega_r}, \tag{3.13}$$

where $\Omega'_r = T(\mathcal{U}'_r(z_0))$, $N_0 = 8(m_1+2m_2)/\sqrt{\pi}m_1$, and $k_0 = m_1^2/16(m_1+2m_2)^2$. It is obvious by (3.9) that (3.6) follows from (3.13).

Next, we turn to the proof of (3.7). Note that by a similar calculation as in (3.12), we have (recall $0<r<1$)

$$\|\partial_x a\|_{L^\infty(Q_r)} \le 4(m_1+2m_2), \quad \|\partial_t a\|_{L^\infty(Q_r)} \le 4m_2/(m_1+2m_2). \tag{3.14}$$

Let us denote $\Pi_\rho := (1-\rho^2, 1)\times(-\rho,\rho)$ for $\rho>0$. Note that if $(t,x)\in\Pi_\rho$, then

$$a(t,x) \ge \frac{1}{2(cr)^2}\left(|\psi(t_0) - \psi(t_0+r)| - |\psi(t_0+r) - \psi(t_0+rt)| - cr^{3/2}|x|\right)^2 \ge \frac{1}{2(cr)^2}\left(m_1 r - m_2 r\rho^2 - cr^{3/2}\rho\right)^2 \ge \frac{1}{2c^2}\left(m_1 - m_2\rho^2 - c\rho\right)^2. \tag{3.15}$$

Fix $\rho_0 = \rho_0(m_1,m_2)\in(0,1/2]$ such that

$$m_1 - m_2\rho_0^2 - c\rho_0 \ge m_1/2 \quad \text{and} \quad \Pi_{\rho_0}\subset\Omega'_r.$$

Then by (3.12) and (3.15), we have

$$2m_1/(m_1+2m_2)^2 \le a(t,x) \le 1, \quad \forall (t,x)\in\Pi_{\rho_0}. \tag{3.16}$$

By (3.14), (3.16), and the interior Schauder estimates, we have

$$|w_x(1,0)| + |w_{xx}(1,0)| + |w_t(1,0)| \le C|w|_{0;\Pi_{\rho_0}}, \tag{3.17}$$

where $C = C(m_1,m_2)$; see e.g. [3]. Now, the estimate (3.7) follows from (3.9), (3.13), and (3.17). The lemma is proved. □

We are ready to prove that $v\in C^{1,2}_{loc}(H_T)$. We define $v_x = 0$ (resp. $v_{xx}=0$, $v_t = 0$) on Γ. By (3.4), it is enough to show that v_x (resp. v_{xx}, v_t) is continuous at each $z_0 = (t_0,x_0)\in\Gamma$. Fix an $r_0 = r_0(z_0)\in(0,1)$ such that $\overline{C}_{r_0}(z_0)\subset D = (0,T)\times(0,\ell)$. Note that for any $z_1\in\Gamma_{r_0/4}(z_0)$ and $r<r_0/4$, we have $C_r(z_1)\subset C_{r_0}(z_0)$. Therefore, by Lemma 3.5

$$|w(\phi(x)+r,x)| \le N_1 r^{-\beta} e^{-k_0/r}|v|_{0;D}, \quad \forall r\in(0,r_0/4) \quad \forall x\in(x_0 - r_0/4, x_0+r_0/4), \tag{3.18}$$

where $w := v_x$ (resp. $w := v_{xx}$, $w := v_t$) and $\beta = -1$ (resp. $\beta = -5/2$, $\beta = -1/2$). On the other hand, note that there is some $\delta = \delta(m_1,m_2)>0$ such that

$$\mathcal{U}_{\delta r_0}(z_0) \subset \{(\phi(x)+r,x)\in\mathbb{R}^2 : 0<r<r_0/4, |x-x_0|<r_0/4\}. \tag{3.19}$$

From (3.18) and (3.19), we find that $\lim_{\rho\to 0}|w|_{0;C_\rho(z_0)} = 0$. The theorem is proved.

4. Generalization of Key lemma

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function satisfying $\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \le M_0$ for some $M_0 \in (0,\infty)$ and denote

$$\Gamma := \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t = \phi(x)\}.$$

For $z = (t,x) \in \mathbb{R}\times\mathbb{R}^n$ and $r > 0$, we shall denote

$$\begin{aligned}
C_r(z) &:= \{(s,y) \in \mathbb{R}\times\mathbb{R}^n : |s-t| < r,\ \max_{1\le k\le n} |y_k - x_k| < (1/2M_0)r\},\\
\mathcal{U}_r(z) &:= C_r(z) \cap \{(s,y) \in \mathbb{R}\times\mathbb{R}^n : s > \phi(y)\},\\
\mathcal{U}'_r(z) &:= \{(s,y) \in \mathcal{U}_r(z) : \max_{1\le k\le n} |y_k - x_k| < (1/4M_0)r\},\\
\Gamma_r(z) &:= C_r(z) \cap \Gamma.
\end{aligned}$$

Theorem 4.1. *Let $z_0 \in \Gamma$ and $r > 0$ be given. Assume that there are numbers $\mu > 1$ and $\Lambda > 0$ such that the coefficients $(a_{ij}(t,x))_{i,j=1}^n$ satisfy*

$$0 \le a_{ij}(t,x)\xi_i\xi_j \le \Lambda\,|\phi(x) - t|^\mu\,|\xi|^2, \quad \forall (t,x) \in C_r(z_0), \quad \forall \xi \in \mathbb{R}^n. \tag{4.2}$$

Let $u \in C^{1,2}_{loc}(\mathcal{U}_r(z_0)) \cap C(\overline{\mathcal{U}}_r(z_0))$ satisfy

$$\begin{cases} Lu := u_t - a_{ij}D_{ij}u = 0 & \text{in} \quad \mathcal{U}_r(z_0),\\ u = 0 & \text{on} \quad \Gamma_r(z_0).\end{cases}$$

Then the following estimate holds.

$$|u|_{0;\mathcal{U}'_r(z_0)} \le N_0 r^{(\mu-1)/2} e^{-k_0 r^{1-\mu}}\,|u|_{0;\mathcal{U}_r(z_0)}, \tag{4.3}$$

where $N_0 = N_0(n,\mu,\Lambda,M_0)$ and $k_0 = k_0(\mu,\Lambda,M_0) > 0$.

Proof. The proof is a slight modification of that in Lemma 2.1. By renormalizing u to $u/\,|u|_{0;\mathcal{U}_r(z_0)}$, we may assume $|u|_{0;\mathcal{U}_r(z_0)} = 1$. Let T be a linear mapping defined by

$$T(t,x) := \left((t-t_0)/r, (x-x_0)/cr^{(1+\mu)/2}\right), \quad \text{where } c := \Lambda^{1/2}(3/2)^{\mu/2}. \tag{4.4}$$

Denote $\Omega_r := T(\mathcal{U}_r(z_0))$, $\Omega'_r := T(\mathcal{U}'_r(z_0))$, $\Sigma_r := T(\Gamma_r(z_0))$, and

$$Q_r := T(C_r(z_0)) = \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : |t| < 1,\ \max_{1\le k\le n}|x_k| < (1/2cM_0)r^{(1-\mu)/2}\}. \tag{4.5}$$

Define the functions $w(t,x)$ and $\tilde{a}_{ij}(t,x)$ on $\overline{\Omega}_r$ and Q_r, respectively, by

$$w(t,x) := u\circ T^{-1}(t,x) = u(t_0 + rt, x_0 + cr^{(1+\mu)/2}x), \tag{4.6}$$

$$\tilde{a}_{ij}(t,x) := (c^2r^\mu)^{-1}\,a_{ij}(t_0 + rt, x_0 + cr^{(1+\mu)/2}x). \tag{4.7}$$

Then $w \in C^{1,2}_{loc}(\Omega_r) \cap C(\overline{\Omega}_r)$ and satisfies

$$\begin{cases} \tilde{L}w := w_t - \tilde{a}_{ij}(t,x)D_{ij}w = 0 & \text{in} \quad \Omega_r,\\ w = 0 & \text{on} \quad \Sigma_r,\end{cases} \tag{4.8}$$

By (4.2) and (4.6), for all $(t,x) \in Q_r$ and $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned}
0 \le \tilde{a}_{ij}(t,x)\xi_i\xi_j &\le \frac{\Lambda}{c^2r^\mu}\left(|\phi(x_0 + cr^{(1+\mu)/2}x) - \phi(x_0)| + r|t|\right)^\mu |\xi|^2\\
&\le \frac{\Lambda}{c^2r^\mu}\left(M_0cr^{(1+\mu)/2}|x| + r|t|\right)^\mu |\xi|^2 \le \frac{\Lambda}{c^2}(3/2)^\mu|\xi|^2 = |\xi|^2.
\end{aligned} \tag{4.9}$$

Let v be given as in (2.4) with $R = (1/2cM_0)r^{(1-\mu)/2}$ and define

$$V(t,x) = V(t,x_1,\ldots,x_n) := \sum_{k=1}^n v(t+1, x_k). \tag{4.10}$$

Then, since $v_{xx} \geq 0$ by (2.6) and $\tilde{a}_{kk} \leq 1$, for all $k = 1, \ldots, n$, by (4.9), we have

$$\begin{aligned} \tilde{L}V = V_t - \tilde{a}_{ij} D_{ij} V &= \sum_{k=1}^{n} (v_t(t, x_k) - \tilde{a}_{kk} v_{xx}(t, x_k)) \\ &\geq \sum_{k=1}^{n} (v_t(t, x_k) - v_{xx}(t, x_k)) = 0 \quad \text{in} \quad Q_r. \end{aligned}$$

Note that by (2.5), $V \geq 1$ on $\partial_x Q_r := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : |t| < 1, |x_k| = R, \forall k = 1, \ldots, n\}$. Also, observe that $\Sigma_r \subset \{(t, x) \in Q_r : |t| < 1/2\}$. Therefore, we have $V \geq |w|$ on the parabolic boundary $\partial_p \Omega_r$ of Ω_r. Then, by the comparison principle, we obtain

$$|w(t, x)| \leq V(t, x), \quad \forall (t, x) \in \Omega_r. \tag{4.11}$$

On the other hand, by (2.6), (2.7), and (4.10), we have (recall $R = (1/2cM_0) r^{(1-\mu)/2}$)

$$V(t, x) \leq (32ncM_0/\sqrt{2\pi}) r^{(\mu-1)/2} e^{-r^{1-\mu}/128c^2 M_0^2} \quad \forall (t, x) \in \Omega'_r. \tag{4.12}$$

We obtain (4.3) by combining (4.6), (4.11), and (4.12). The theorem is proved. □

Theorem 4.13. *Let $\bar{z}_0 \in \Gamma$ and $R > 0$ be given. Assume that there are numbers $\mu > 1$ and $\lambda, \Lambda, M_1 > 0$ such that the coefficients $(a_{ij}(t, x))_{i,j=1}^n$ satisfy*

$$\lambda \, |\phi(x) - t|^{\mu} \, |\xi|^2 \leq a_{ij}(t, x) \xi_i \xi_j \leq \Lambda \, |\phi(x) - t|^{\mu} \, |\xi|^2, \quad \forall (t, x) \in C_R(\bar{z}_0), \quad \forall \xi \in \mathbb{R}^n, \tag{4.14}$$

$$|\nabla_{t,x} a_{ij}(t, x)| \leq M_1 \, |\phi(x) - t|^{\mu - 1}, \quad \textit{for a.e.} \ (t, x) \in C_R(\bar{z}_0). \tag{4.15}$$

Suppose $u \in C^{1+\alpha/2, 2+\alpha}_{loc}(\mathcal{U}_R(\bar{z}_0)) \cap C(\overline{\mathcal{U}}_R(\bar{z}_0))$, for some $\alpha \in (0, 1)$, and satisfies

$$\begin{cases} Lu := u_t - a_{ij} D_{ij} u = 0 & \textit{in} \quad \mathcal{U}_R(\bar{z}_0), \\ u = 0 & \textit{on} \quad \Gamma_R(\bar{z}_0). \end{cases}$$

Then if we extend $u \equiv 0$ in $C_R(\bar{z}_0) \setminus \mathcal{U}_R(\bar{z}_0)$, we have $u \in C^{1,2}_{loc}(C_{R/2}(\bar{z}_0))$

Proof. Let $z_0 = (t_0, x_0) = (\phi(x_0), x_0) \in \Gamma_{R/2}(\bar{z}_0)$ and let $0 < r < \min(1, R/2)$ so that $r < 1$ and $C_r(z_0) \subset C_R(\bar{z}_0)$. Then, by (4.3) of Theorem 4.1 we find

$$|u|_{0; \mathcal{U}'_r(z_0)} \leq N_0 r^{(\mu-1)/2} e^{-k_0 r^{1-\mu}} \, |u|_{0; \mathcal{U}_R(\bar{z}_0)}, \tag{4.16}$$

Let T, Q_r, $w(t, x)$, and $\tilde{a}_{ij}(t, x)$ be defined as in (4.4) – (4.7). Then, by (4.15) we have

$$\|\nabla_{t,x} \tilde{a}_{ij}\|_{L^\infty(Q_r)} \leq CM_1, \quad \text{where } C = C(\Lambda, \mu). \tag{4.17}$$

Denote $\Pi_\rho := (1 - \rho^2, 1) \times (-\rho, \rho)^n$. Note that if $(t, x) \in \Pi_\rho$, then we have

$$|\phi(x_0 + cr^{(1+\mu)/2} x) - (t_0 + rt)| \geq |r(1 - \rho^2) - M_0 c r^{(1+\mu)/2} \rho| \geq r|1 - \rho^2 - M_0 c \rho|.$$

Let us fix a number $\rho_0 = \rho_0(\mu, \Lambda, M_0) \in (0, 1/2)$ such that $|1 - \rho^2 - M_0 c \rho| \geq (1/2)^{1/\mu}$ and $\Pi_{\rho_0} \subset \Omega'_r$. Then, it follows from (4.7) and (4.14) that

$$\tilde{a}_{ij}(t, x) \xi_i \xi_j \geq (\lambda/2c^2) |\xi|^2 = (\lambda/\Lambda) 2^{\mu-1} 3^{-\mu} |\xi|^2. \tag{4.18}$$

Then by (4.17), (4.9), (4.18), and the interior Schauder estimate, we have

$$|D_x w(1, 0)| + |D_x^2 w(1, 0)| + |w_t(1, 0)| \leq C |w|_{0; \Pi_{\rho_0}}, \tag{4.19}$$

where $C = C(n, \alpha, \mu, \lambda, \Lambda, M_0, M_1)$. Therefore, by using (4.6) and (4.16), we conclude

$$\begin{aligned} r^{(1+\mu)/2} |D_x u(t_0 + r, x_0)| + r^{1+\mu} |D_x^2 u(t_0 + r, x_0)| + r |u_t(t_0 + r, x_0)| \\ \leq N_1 r^{(\mu-1)/2} e^{-k_0 r^{1-\mu}} |u|_{0; \mathcal{U}_R(\bar{z}_0)}, \end{aligned} \tag{4.20}$$

where $N_1 = N_1(n, \alpha, \mu, \lambda, \Lambda, M_0, M_1)$. Finally, by using (4.20) instead of (3.7) and proceeding similarly as in the proof of Theorem 1.8, we see that $u \in C^{1,2}_{loc}(C_{R/2}(\bar{z}_0))$. This completes the proof. $\square$

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